

# Distance Graphs and $T$ -Coloring

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We discuss relationships among  $T$ -colorings of graphs and chromatic numbers, fractional chromatic numbers, and circular chromatic numbers of distance graphs. We first prove that for any finite integral set  $T$  that contains 0, the asymptotic  $T$ -coloring ratio  $R(T)$  is equal to the fractional chromatic number of the distance graph  $G(Z, D)$ , where  $D = T - \{0\}$ . This fact is then used to study the distance graphs with distance sets of the form  $D_{m,k} = \{1, 2, \dots, m\} - \{k\}$ . The chromatic numbers and the fractional chromatic numbers of  $G(Z, D_{m,k})$  are determined for all values of  $m$  and  $k$ . Furthermore, circular chromatic numbers of  $G(Z, D_{m,k})$  for some special values of  $m$  and  $k$  are obtained. © 1999 Academic Press

## 1. INTRODUCTION

The  $T$ -coloring problem was formulated by Hale [18] as a model for the channel assignment problem, in which an integer broadcast channel is

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assigned to each of several locations so that interference among nearby locations is avoided. Interference is modeled by a non-negative integral set  $T$  containing 0 (called a  $T$ -set) as forbidden channel separations. One can construct a graph  $G = (V, E)$  such that each vertex represents a location; two vertices are adjacent if their corresponding locations are nearby. Thereafter a valid channel assignment or  $T$ -coloring is a mapping  $f$  from the vertex set of  $V$  of  $G$  to the set of non-negative integers  $\{0, 1, 2, \dots\}$  such that  $|f(x) - f(y)| \notin T$  whenever  $xy \in E$ . The *span* of a  $T$ -coloring  $f$  is the difference between the largest and the smallest numbers in  $f(V)$ , i.e.,  $\max\{|f(u) - f(v)| : u, v \in V\}$ . Given  $T$  and  $G$ , the  $T$ -span of  $G$ , denoted by  $\text{sp}_T(G)$ , is the minimum span among all  $T$ -colorings of  $G$ .

$T$ -coloring has been extensively studied in the literature (see [4, 5, 16, 23–26, 30–32, 35]). Let  $\sigma_n$  denote  $\text{sp}_T(K_n)$ , where  $K_n$  is a complete graph with  $n$  vertices. Griggs and Liu [16] proved that the *difference optimum sequence*,  $\Delta\sigma = (\sigma_{n+1} - \sigma_n)_{n=1}^\infty$ , is eventually periodic. This implies that for any  $T$ -set, the *asymptotic  $T$ -coloring ratio*

$$R(T) := \lim_{n \rightarrow \infty} \frac{\sigma_n}{n}$$

exists and is a rational number. This result was also proven by Rabinowitz and Proulx [30] and Cantor and Gordon [1] by different approaches.

The notion of distance graphs originated with the plane-coloring problem: What is the smallest number of colors needed to color all points of a Euclidean plane such that points at unit distances are colored with different colors. It is well known that four colors are necessary [28] and seven colors are sufficient [17]. However, the exact number of colors needed remains unknown (see [6]). Motivated by this problem, Eggleton [10] made the following generalization. Suppose  $S$  is a subset of a metric space  $\mathcal{M}$  with metric  $d$ , and  $D$  is a set of positive real numbers. The *distance graph*  $G(S, D)$  with *distance set*  $D$  is the graph with vertex set  $S$  and edge set  $\{xy : d(x, y) \in D\}$ . The objective is to determine  $\chi(S, D)$ , the chromatic number of  $G(S, D)$ . Note that the plane-coloring problem introduced above is equivalent to finding  $\chi(R^2, \{1\})$ .

Let  $D$  be a set of positive integers (called a  $D$ -set). The distance graph to be studied in this article is  $G(Z, D)$ , which has  $Z$  as the vertex set and  $\{uv : |u - v| \in D\}$  as the edge set. The problem of finding  $\chi(Z, D)$  for different  $D$ -sets has been studied extensively (see [2, 7, 9, 11–14, 21, 37–41]).

A *fractional coloring* of a graph  $G$  is a mapping  $c$  from  $\mathcal{I}(G)$ , the set of all independent sets of  $G$ , to the interval  $[0, 1]$  such that  $\sum_{x \in I \in \mathcal{I}(G)} c(I) \geq 1$  for all vertices  $x$  in  $G$ . The *fractional chromatic number*  $\chi_f(G)$  of  $G$  is the infimum of the value  $\sum_{I \in \mathcal{I}(G)} c(I)$  of a fractional coloring  $c$  of  $G$  ([15, 22, 33, 34]).

For a given  $T$ -set, letting  $D = T - \{0\}$ , Liu [26] proved that the asymptotic  $T$ -coloring ratio  $R(T)$  is a lower bound of  $\chi(Z, D)$ . Hence,  $T$ -colorings and distance graphs are closely related. We shall explore further relationships between these two concepts. In Section 2, we prove that for any  $T$ -set,  $R(T)$  is equal to the fractional chromatic number of the distance graph  $G(Z, D)$ , if  $D = T - \{0\}$ . This relationship provides new insights concerning the parameter  $R(T)$ , and can be used to simplify the proofs of some known results regarding  $R(T)$ .

Section 3 focuses on the family of distance graphs with  $D$ -sets of the form  $D_{m,k} = \{1, 2, \dots, m\} - \{k\}$ . The chromatic numbers of such distance graphs, denoted as  $\chi(Z, D_{m,k})$ , have been investigated in the following articles. In [11], Eggleton, Erdős and Skilton obtained the solution for  $k = 1$ , and partial solutions for  $k = 2$ :  $\chi(Z, D_{m,1}) = \lfloor (m+3)/2 \rfloor$  for any  $m \geq 2$ ,  $\chi(Z, D_{m,2}) = \lfloor (m+4)/2 \rfloor$  when  $m \not\equiv 3 \pmod{4}$ , and  $\lfloor (m+3)/2 \rfloor \leq \chi(Z, D_{m,2}) \leq \lfloor (m+5)/2 \rfloor$  for any  $m \geq 4$  with  $m \equiv 3 \pmod{4}$ . For  $3 \leq k < m$ , the same authors provided the bounds

$$\max \left\{ k, \left\lfloor \frac{1}{2} \left( \frac{m}{k-1} + 1 \right) \right\rfloor t \right\} \leq \chi(Z, D_{m,k}) \leq \min \left\{ m, \left\lfloor \frac{1}{2} \left( \frac{m}{k} + 3 \right) \right\rfloor k \right\},$$

where  $t = 2$  if  $k = 3$ , and  $t = k - 2$  if  $k \geq 4$ . The same result for the case  $k = 1$  was also proven by Kemnitz and Kolberg in [21] by a different approach. The lower bound of  $\chi(Z, D_{m,k})$  in the above has been improved to  $\lceil (m+k+1)/2 \rceil$  by Liu ([26]), who also showed that the new bound is sharp for all pairs of integers  $(m, k)$  where  $k$  is odd. Furthermore, complete solutions for  $k = 2$  and 4, and partial solutions for other even integers  $k$  are given in [26].

The main results of this paper are complete solutions for the chromatic numbers and the fractional chromatic numbers of the distance graphs  $G(Z, D_{m,k})$  for all values  $m$  and  $k$ . These results are also applied to study the circular chromatic number of distance graphs.

Suppose  $k$  and  $d$  are positive integers such that  $k \geq 2d$ . A  $(k, d)$ -coloring of a graph  $G = (V, E)$  is a mapping  $c$  from  $V$  to  $\{0, 1, \dots, k-1\}$  such that  $\|c(x) - c(y)\|_k \geq d$  for any edge  $xy$  in  $G$ , where  $\|a\|_k = \min\{a, k-a\}$ . The circular chromatic number  $\chi_c(G)$  of  $G$  is the infimum of  $k/d$  for all  $(k, d)$ -colorings of  $G$ . The circular chromatic number is also known as the *star-chromatic number* in the literature (see [36, 42, 43]).

For any graph  $G$ , it is well known that

$$\max \left\{ \omega(G), \frac{|V(G)|}{\alpha(G)} \right\} \leq \chi_f(G) \leq \chi_c(G) \leq \lceil \chi_c(G) \rceil = \chi(G). \quad (*)$$

The parameters involved in (\*) for distance graphs are explored in this paper. For simplicity, let  $\omega(Z, D)$ ,  $\alpha(Z, D)$ ,  $\chi_f(Z, D)$ , and  $\chi_c(Z, D)$  denote the clique number, the independence number, the fractional chromatic number, and the circular chromatic number of  $G(Z, D)$ , respectively.

## 2. RELATIONSHIPS BETWEEN $R(T)$ AND $\chi_f(Z, D)$

This section shows that for any  $T$ -set, the asymptotic  $T$ -coloring ratio  $R(T)$  is equal to the fractional chromatic number of the distance graph  $G(Z, D)$ , if  $D = T - \{0\}$ . Based upon this result, we give simpler and different proofs of some known facts regarding  $R(T)$ .

**THEOREM 1.** *For any finite  $T$ -set, if  $D = T - \{0\}$ , then  $R(T) = \chi_f(Z, D)$ .*

*Proof.* Suppose  $c$  is an optimal  $T$ -coloring of  $K_n$ , where  $0 = c(1) < c(2) < \dots < c(n) = \sigma_n$ . Let  $m = 1 + \max_{d \in D} d$ . For  $1 \leq i \leq c(n) + m$ , let

$$I_i = \{j \in Z : j \equiv i + c(k) \pmod{c(n) + m} \text{ for some } k \text{ with } 1 \leq k \leq n\}.$$

It is straightforward to verify that each  $I_i$  is an independent set in  $G(Z, D)$ , and that every integer belongs to exactly  $n$  of the independent sets  $I_i$  ( $1 \leq i \leq c(n) + m$ ). Define a mapping  $c'$ :  $\mathcal{I}(G(Z, D)) \rightarrow [0, 1]$  as

$$c'(I) = \begin{cases} \frac{1}{n}, & \text{if } I = I_i \text{ for } 1 \leq i \leq c(n) + m; \\ 0, & \text{otherwise.} \end{cases}$$

Then  $c'$  is a fractional coloring of  $G(Z, D)$ . This implies that for any positive integer  $n$ ,  $\chi_f(Z, D) \leq (c(n) + m)/n$ . Hence, we have

$$\chi_f(Z, D) \leq \lim_{n \rightarrow \infty} \frac{c(n) + m}{n} = \lim_{n \rightarrow \infty} \frac{\sigma_n}{n} = R(T).$$

To show  $\chi_f(Z, D) \geq R(T)$ , let  $G_n$  be the subgraph of  $G(Z, D)$  induced by the vertex set  $\{0, 1, 2, \dots, c(n)\}$ ; i.e.,  $G_n = G(\{0, 1, 2, \dots, c(n)\}, D)$ . Then the set of vertices  $\{c(1), c(2), \dots, c(n)\}$  is a maximum independent set in  $G_n$ . So, for any positive integer  $n$ , we have

$$\chi_f(Z, D) \geq \chi_f(G_n) \geq \frac{|V(G_n)|}{\alpha(G_n)} = \frac{c(n) + 1}{n}.$$

This implies

$$\chi_f(Z, D) \geq \lim_{n \rightarrow \infty} \frac{c(n) + 1}{n} = \lim_{n \rightarrow \infty} \frac{\sigma_n}{n} = R(T).$$

Therefore,  $R(T) = \chi_f(Z, D)$ .

Q.E.D

The theorem above provides new insights into the asymptotic  $T$ -coloring ratio  $R(T)$ . Some previous results concerning  $R(T)$  can be obtained from this approach. For example, it is well known that  $R(T) \geq 2$ , provided  $T \neq \{0\}$  (see [1, 16, 30]). This is straightforward when we consider fractional chromatic numbers, since the fractional chromatic number of any non-trivial graph is at least 2. Moreover, for a non-trivial graph  $G$ ,  $\chi_f(G) = 2$  if and only if  $G$  is bipartite. Since  $G(Z, D)$  is bipartite if and only if  $D$  contains no even integers (assuming that  $\gcd(T) = 1$ ), we have the following result.

**COROLLARY 2.** *For any  $T$ -set with  $\gcd(T) = 1$ ,  $R(T) = 2$  if and only if  $T$  contains only odd integers except 0.*

Theorem 1 can also be applied to some other known results about  $R(T)$  which are closely related to an earlier number theory problem, namely, sequences with missing differences. Given a  $T$ -set, a  $T$ -sequence is an increasing sequence  $S$  of nonnegative integers such that  $x - y \notin T$  for any  $x, y \in S$ . Motzkin [29] proposed studying the supremum  $\mu(T)$  of the asymptotic upper densities of these sequences  $S$ . Cantor and Gordon [1] determined the exact values of  $\mu(T)$  when  $|T| = 2$  and 3. Haralambis [19] gave partial solutions when  $|T| = 4$  or 5. It is known that  $\mu(T)$  is equal to the reciprocal of  $R(T)$  (see [16]). Therefore results on sequences with missing differences can be applied to  $T$ -colorings. Cantor and Gordon [1] and Rabinowitz and Proulx [30] proved that if  $T = \{0, a, b\}$ ,  $\gcd(a, b) = 1$ , and  $a$  and  $b$  are of different parity, then  $R(T) = 2(a+b)/(a+b-1)$ . The original argument in proving the inequality  $R(T) \geq 2(a+b)/(a+b-1)$  in [1] was quite complicated. However, applying some facts about fractional chromatic number, one can obtain the following simpler proof. Since  $a$  and  $b$  are of opposite parity,  $G(Z, D)$  with  $D = \{a, b\}$  contains an odd cycle  $C_{a+b}$ . As  $\chi_f(C_{2m+1}) = 2 + 1/m$ , it follows that  $\chi_f(Z, D) \geq \chi_f(C_{a+b}) = 2(a+b)/(a+b-1)$ . Thus, by Theorem 1,  $R(T) \geq 2(a+b)/(a+b-1)$ . Indeed, combining Theorem 1 with the fact that  $\chi_f(H) \leq \chi_f(G)$  if  $H$  is a subgraph of  $G$ , the following is obvious.

**COROLLARY 3.** *If  $H$  is a subgraph of the distance graph  $G(Z, D)$ , then  $\chi_f(H) \leq R(T)$ , where  $T = D \cup \{0\}$ .*

3. VALUES OF  $\chi_f(Z, D_{m,k})$ ,  $\chi(Z, D_{m,k})$ , AND  $\chi_c(Z, D_{m,k})$ 

In this section, we first calculate  $\chi_f(Z, D_{m,k})$ , which, according to (\*), is a lower bound of  $\chi(Z, D_{m,k})$ . We then determine  $\chi(Z, D_{m,k})$  for all values of  $m$  and  $k$ . Using this approach, circular chromatic numbers of  $G(Z, D_{m,k})$  for special values of  $m$  and  $k$  are obtained as well.

The values of  $R(T)$  as  $T = D_{m,k} \cup \{0\}$  are given in [26]. Thus, the following two results can also be obtained by using Theorem 1. Here we include methods of calculating  $\chi_f(Z, D_{m,k})$  directly.

**THEOREM 4.** *If  $2k > m$ , then*

$$\omega(Z, D_{m,k}) = \chi_f(Z, D_{m,k}) = \chi_c(Z, D_{m,k}) = \chi(Z, D_{m,k}) = k.$$

*Proof.* Since the set of vertices  $\{1, 2, \dots, k\}$  forms a clique in  $G(Z, D_{m,k})$ ,  $k \leq \omega(Z, D_{m,k})$ .

By (\*), it is sufficient to show  $\chi(Z, D_{m,k}) \leq k$ . Define a vertex-coloring  $f$  on  $Z$  as  $f(i) = (i \bmod k)$ . Then  $f$  is a proper coloring, since  $D_{m,k}$  contains no multiple of  $k$ . Q.E.D

**THEOREM 5.** *If  $2k \leq m$ , then  $\chi_f(Z, D_{m,k}) = (m + k + 1)/2$ .*

*Proof.* For any  $i$  with  $0 \leq i \leq m + k$ ,  $I_i = \{j \in Z : j - i \equiv 0 \text{ or } k \pmod{m + k + 1}\}$  is an independent set in  $G(Z, D_{m,k})$ . Furthermore, each integer is contained in exactly two such independent sets. Define a mapping  $c: \mathcal{I}(G(Z, D_{m,k})) \rightarrow [0, 1]$  as

$$c(I) = \begin{cases} \frac{1}{2}, & \text{if } I = I_i \text{ for } 0 \leq i \leq m + k; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that  $c$  is a fractional coloring of  $G(Z, D_{m,k})$ . Thus,  $\chi_f(Z, D_{m,k}) \leq (m + k + 1)/2$ .

Since  $\alpha(\{0, 1, \dots, m + k\}, D_{m,k}) = 2$  for  $2k \leq m$ , by (\*),  $\chi_f(\{0, 1, \dots, m + k\}, D_{m,k}) \geq (m + k + 1)/2$ . Therefore, the proof is complete. Q.E.D

Because  $\chi(G)$  is an integer, Theorem 5 and (\*) imply the following:

**COROLLARY 6** [26]. *If  $2k \leq m$  then  $\chi(Z, D_{m,k}) \geq \lceil (m + k + 1)/2 \rceil$ .*

We are now in a position to give the complete solutions to  $\chi(Z, D_{m,k})$  for all values of  $m$  and  $k$ . This is accomplished in the next two results. As will be shown,  $\chi(Z, D_{m,k})$  is either  $\lceil (m + k + 1)/2 \rceil$  or  $\lceil (m + k + 1)/2 \rceil + 1$ .

LEMMA 7. Suppose  $2k \leq m$ . Write  $m+k+1 = 2^r m'$  and  $k = 2^s k'$ , where  $r$  and  $s$  are non-negative integers and  $m'$  and  $k'$  are odd integers. If  $1 \leq r \leq s$ , then  $\chi(Z, D_{m,k}) > (m+k+1)/2$ .

*Proof.* Since  $1 \leq r$ ,  $m+k+1$  is even. Assume to the contrary that  $\chi(Z, D_{m,k}) \leq (m+k+1)/2$ . By Corollary 6,  $\chi(Z, D_{m,k}) = (m+k+1)/2$ . Color  $G(Z, D_{m,k})$  by using  $(m+k+1)/2$  colors.

For each integer  $i$ , consider the subgraph of  $G(Z, D_{m,k})$  induced by the  $m+k+1$  vertices  $\{i, i+1, \dots, i+m+k\}$ . This subgraph has independence number 2. Hence, each of  $(m+k+1)/2$  colors is used at most, and hence exactly, twice in this subgraph. Thus, each color is used exactly twice in any consecutive  $m+k+1$  vertices. Consequently, vertices  $i$  and  $i+m+k+1$  have the same colors for all  $i \in Z$ . Therefore, for each  $i \in S := \{0, 1, \dots, m+k\}$ , the only possible vertices in  $S$  having the same color as  $i$  are  $i+k$  and  $i-k \pmod{m+k+1}$ .

Consider the circulant graph  $C(m+k+1, k)$ , with vertex set  $S$  and in which vertex  $i$  is adjacent to vertex  $j$  if and only if  $j \equiv i+k$  or  $i-k \pmod{m+k+1}$ . It follows from the discussion in the preceding paragraph that two vertices  $x$  and  $y$  of  $S$  have the same color only if  $xy$  is an edge of the circulant graph  $C(m+k+1, k)$ . Since the intersection of each color class with  $S$  contains exactly two vertices, the coloring induces a perfect matching of  $C(m+k+1, k)$ . However,  $C(m+k+1, k)$  is the disjoint union of  $d$  cycles of length  $(m+k+1)/d$ , where  $d = \gcd(m+k+1, k)$ . Since  $C(m+k+1, k)$  has a perfect matching, each cycle has an even length. This implies that  $r > s$ , contrary to the assumption  $r \leq s$ . Q.E.D

The next theorem determines the chromatic number  $\chi(Z, D_{m,k})$  for all values of  $m$  and  $k$ . Incidentally, it also shows that the converse of Lemma 7 is true.

THEOREM 8. Suppose  $2k \leq m$ . Write  $m+k+1 = 2^r m'$  and  $k = 2^s k'$ , where  $r$  and  $s$  are non-negative integers and  $m'$  and  $k'$  are odd integers. Then

$$\chi(Z, D_{m,k}) = \begin{cases} \frac{m+k+1}{2}, & \text{if } r > s; \\ \left\lfloor \frac{m+k+2}{2} \right\rfloor, & \text{otherwise.} \end{cases}$$

*Proof.* It follows from Corollary 6 and Lemma 7 that if  $r > s$ , then  $\chi(Z, D_{m,k}) \geq (m+k+1)/2$ ; if  $r \leq s$ , then  $\chi(Z, D_{m,k}) \geq \lceil (m+k+2)/2 \rceil$ . Therefore it suffices to show that  $G(Z, D_{m,k})$  is  $(m+k+1)/2$ -colorable, if  $r > s$ ; and  $G(Z, D_{m,k})$  is  $\lceil (m+k+2)/2 \rceil$ -colorable, if  $r \leq s$ . It is known that  $\chi(Z, D) = \chi(Z^+ \cup \{0\}, D)$  [14]. Therefore, it is sufficient to find a proper

coloring for the subgraph of  $G(Z, D_{m,k})$  induced by all non-negative integers.

We first decompose  $k$  into the sum of an odd number of integers,  $k = a_1 + a_2 + \cdots + a_p$ , as follows:

*Case 1.* For  $r > s$  or  $s = 0$ , let  $p = k'$  and  $a_j = 2^s$  for  $1 \leq j \leq p$ .

*Case 2.* For  $r \leq s \neq 0$ , let  $p = k - 1$  and  $a_j = 1$  for  $1 \leq j < p$  and  $a_p = 2$ .

Next, partition the set  $Z$  into consecutive blocks of sizes  $a_1, a_2, \dots, a_p$  periodically. Then “pre-color” the blocks, alternating RED and BLUE. We call a vertex RED (or BLUE) if it falls within a RED (or BLUE) block.

Define a coloring  $f$  on the vertices of all non-negative integers of  $G(Z, D_{m,k})$  according to the following three rules:

(R1)  $f(i) = i$ , if  $0 \leq i \leq k - 1$ ;

(R2)  $f(i) = f(i - k)$ , if  $i$  is BLUE and  $i \geq k$ ;

(R3)  $f(i) =$  the smallest non-negative integer that has not been used as a color in the  $m$  vertices preceding  $i$ , if  $i$  is RED and  $i \geq k$ .

To show that  $f$  is a proper coloring, we claim that for any vertex  $i$ ,  $f(i) \neq f(j)$  for all  $j \neq i - k$  with  $i - m \leq j < i$ . It is easy to see that the claim is true when (R1) or (R3) is performed. Suppose (R2) is executed, i.e.,  $i$  is BLUE,  $i \geq k$ , and  $f(i) = f(i - k)$ . Since  $k$  is divided into an odd number of blocks,  $i - k$  is a RED vertex. By (R1) or (R3),  $f(i - k)$  is different from any of the colors of the  $m$  vertices preceding  $i - k$ . Thus, it is sufficient to show that  $f(i) \neq f(j)$  for all  $j$  with  $i - k < j < i$ .

If  $j$  is RED, by (R1) or (R3),  $f(j) \neq f(i - k)$  and so  $f(i) \neq f(j)$ . If  $j$  is BLUE, by (R1) or (R2),  $f(j) = f(j - k)$ . One has  $i - k - m < j - k < i - k$  (because  $2k \leq m$ ), so  $f(i - k) \neq f(j - k)$ . This implies  $f(i) \neq f(j)$ .

To complete the proof of the theorem, it is sufficient to show that  $f$  is an  $((m + k + 1)/2)$ -coloring if  $r > s$ ; and  $f$  is an  $\lceil (m + k + 2)/2 \rceil$ -coloring if  $r \leq s$ . One can accomplish this by counting the number of colors that have been used for the  $m$  vertices preceding a RED vertex  $i$  for which  $i \geq k$ . The first  $k$  vertices need at most  $k$  colors. For the remaining  $m - k$  vertices, only those RED vertices need new colors.

If  $r > s$ , then  $m - k + 1$  is a multiple of  $2^{s+1}$ . Any consecutive  $2^{s+1}$  vertices have  $2^s$  BLUE vertices and  $2^s$  RED ones, so there are exactly  $(m - k - 1)/2$  RED vertices in the remaining  $m - k$  vertices. Therefore, the total number of colors used in  $f$  is at most  $k + (m - k - 1)/2 + 1 = (m + k + 1)/2$ .

If  $r \leq s$  (with  $s = 0$  in Case 1 and  $s \neq 0$  in Case 2), then there are at most  $\lceil (m - k)/2 \rceil$  RED vertices in the remaining  $m - k$  vertices. Thus, the total number of colors used in  $f$  is at most  $k + \lceil (m - k)/2 \rceil + 1 = \lceil (m + k + 2)/2 \rceil$ . This completes the proof of Theorem 8.



We now present the following two results concerning the circular chromatic number of  $G(Z, D_{m,k})$ . The first one follows from (\*) and Theorems 5 and 8.

**COROLLARY 9.** *Suppose  $2k \leq m$ . Write  $m+k+1=2^r m'$  and  $k=2^s k'$ , where  $r$  and  $s$  are non-negative integers and  $m'$  and  $k'$  are odd integers. If  $r > s$ , then  $\chi_f(Z, D_{m,k}) = \chi_c(Z, D_{m,k}) = \chi(Z, D_{m,k}) = (m+k+1)/2$ .*

**THEOREM 10.** *If  $2k \leq m$  and  $k$  is relatively prime to  $m+k+1$ , then  $\chi_f(Z, D_{m,k}) = \chi_c(Z, D_{m,k}) = (m+k+1)/2$ .*

*Proof.* Since  $k$  is relatively prime to  $m+k+1$ , there exists an integer  $n$  such that  $nk \equiv 1 \pmod{m+k+1}$ . Consider the mapping  $c$  defined by  $c(i) = (in \pmod{m+k+1})$  for all  $i \in Z$ . For any edge  $ij$  in  $G(Z, D_{m,k})$ , we shall prove that  $\|c(i) - c(j)\|_{m+k+1} \geq 2$ . Suppose to the contrary, that  $\|c(i) - c(j)\|_{m+k+1} \leq 1$ ; i.e.,  $c(i) - c(j) \equiv 0$  or  $1$  or  $-1 \pmod{m+k+1}$ . Then  $i - j \equiv 0$  or  $k$  or  $-k \pmod{m+k+1}$ , which contradicts the fact that  $i$  is adjacent to  $j$ . Thus  $c$  is an  $(m+k+1, 2)$ -coloring of  $G(Z, D_{m,k})$ . This along with Theorem 5 and (\*) implies the theorem. Q.E.D

*Remarks.* Many new results related to this topic have been obtained since the submission of this paper. In [3], the circular chromatic numbers of all the graphs  $G(Z, D_{m,k})$  are determined. The chromatic number, circular chromatic number and fractional chromatic number of distance graphs with distance sets of the form  $D_{m,k,s} = \{1, 2, \dots, m\} - \{k, 2k, \dots, sk\}$  have been studied in [8, 20, 27, 44]. (Accordingly, the distance graphs discussed in this paper are  $G(Z, D_{m,k,1})$ .) In [27], the chromatic numbers of all the graphs  $G(Z, D_{m,k,2})$  are determined. The same paper also determined the fractional chromatic numbers of all the graphs  $G(Z, D_{m,k,s})$ . In [8], the following was proved:

$$\lceil (m+sk+1)/(s+1) \rceil \leq \chi(G(Z, D_{m,ks})) \leq \lceil (m+sk+1)/(s+1) \rceil + 1.$$

Moreover, both the upper bound and the lower bound are attainable. Then in [20], the chromatic numbers of all the graphs  $G(Z, D_{m,k,s})$  are completely determined. Finally and most recently, [44] determines the circular chromatic numbers of all the graphs  $G(Z, D_{m,k,s})$ .

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